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A variational approach to magneto-elastic buckling problems for systems of ferromagnetic or superconducting beams

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Abstract. Based upon a variational principle derived in a preceding paper, expressions for the magneto-elastic buckling values for ferromagnetic or superconducting systems are given. These relations are evaluated for systems of slender beams. Explicit buckling values are calculated for a single ferromagnetic or superconducting beam of arbitrary cross-section, and for systems of two parallel ferromagnetic or superconducting rods. In the analysis needed for the calculation of the intermediate (i.e., rigid-body) and the perturbed magnetic fields, an intensive use of methods inherent in the theory of complex functions is made. In conclusion our results for a set of two superconducting rods are compared with the results of a mathematically less complicated, but also less rigorous, theory.

1. Introduction

In [1] the authors derived an explicit relation for a magneto-elastic buckling value by way of a variational principle. This relation was accompanied by equations and boundary conditions for both the intermediate (i.e. pre-buckled or rigid-body) fields and for the perturbed (due to buckling) fields. These fields must be solved first, and then mere substitution of the results into the expression for the buckling value immediately yields an explicit value for the critical or buckling field. In [1] detailed evaluations were given for (i) soft ferromagnetic bodies, and (ii) superconductors.

We start here with recapitulating the main results of [1]. Firstly, for a soft ferromagnetic body in vacuum placed in a uniform field of field strength B_0 one has for the critical value of B_0 the relation (cf. [1], (6.22); for the definitions of the symbols we refer to [1])

$$\frac{\mu_0 \mathrm{E}}{B_0^2} = \left\{\int_{\partial G}\left[(\psi + B_k u_k)\frac{\partial\psi}{\partial N} + B_i u_i \frac{\partial}{\partial N}(\psi + B_k u_k) - B_k u_k B_j u_{j,i} N_i \right.\right.$$
$$\left.\left. + \tfrac{1}{2}B_k B_k (u_{j,j}u_i - u_{i,j}u_j)N_i\right]\mathrm{dS} - \int_{G^-} T_{jk}u_{i,k}u_{i,j}\,\mathrm{dV}\right\}$$
$$\times\left\{\frac{1}{1+\nu}\int_{G^-}\left[\frac{\nu}{1-2\nu}e_{kk}e_{ll} + e_{kl}e_{kl}\right]\mathrm{dV}\right\}^{-1}. \tag{1.1}$$

In this expression $\mathbf{B}$ and T are the normalized magnetic induction in the vacuum G^+ and the normalized stress tensor in the rigid-body state, which have to satisfy (cf. [1], (6.18)–(6.21))

$$\begin{gathered}\operatorname{div}\mathbf{B} = 0,\quad \operatorname{curl}\mathbf{B} = \mathbf{0},\quad x\in G^+;\qquad \mathbf{B}\times\mathbf{N} = \mathbf{0},\quad x\in\partial G;\\ \int_{\partial G}(\mathbf{B},\mathbf{N})\,\mathrm{dS} = 0;\qquad \mathbf{B}\to\mathbf{B}_0/B_0,\quad |\mathbf{x}|\to\infty,\end{gathered} \tag{1.2}$$

and

$$T_{ij,j} = 0, \quad \mathbf{x} \in G^-; \qquad T_{ij}N_j = \tfrac{1}{2}(\mathbf{B}, \mathbf{B})N_i, \quad \mathbf{x} \in \partial G. \tag{1.3}$$

Note that T is not completely determined by (1.3), but this will do for our purposes. Moreover, since we have identified the intermediate state with the rigid-body state, there is no need anymore to distinguish between Lagrange and Euler coordinates.

The field ψ, occurring in (1.1), is the normalized perturbed magnetic potential, due to the deflection $\mathbf{u}$ in buckling. For ψ, we have derived in [1] the relations (cf. [1], (6.10), (6.14))

$$\begin{aligned} &\Delta\psi = \psi_{,ii} = 0, \quad \mathbf{x} \in G^+; \qquad \psi + B_k u_k = \psi_0, \quad \mathbf{x} \in \partial G; \\ &\int_{\partial G} \frac{\partial \psi}{\partial N}\,\mathrm{dS} = 0; \qquad \psi \to 0, \quad |\mathbf{x}| \to \infty. \end{aligned} \tag{1.4}$$

The displacement field $\mathbf{u}$ must be chosen in such a way that it constitutes a reasonable representation for the deflection in buckling for the (mostly slender) body under consideration. Clearly, this choice can only be made after the shape of the body (e.g., a plate or a beam) is known. In the next section this will be made explicit for the case of a slender beam. The linear deformations e_{ij} are related to u_i by

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{1.5}$$

Whenever we can succeed in solving (1.2)–(1.4) and make an acceptable choice for u_i, we only have to substitute the results in (1.1) to obtain a numerical value for the buckling field magnitude (in this case B_0). It is this procedure that we shall follow in this paper.

Secondly, we proceed with the recapitulation of the analogous results for a superconducting structure with total electric current I_0. For the critical current we have derived (cf. [1], (7.18))

$$\begin{aligned} \frac{\mathrm{E}}{\mu_0 I_0^2} = \Bigg\{ &\int_{\partial G} [\psi(B_j u_{i,j} - B_{i,j}u_j) + B_k B_{k,j} u_i u_j - e_{ijm} B_m A_{j,kl} u_k u_l \\ &+ 2B_k(e_{ijk}u_l - e_{ljk}u_i)(A_{j,m}u_m)_{,l} + \tfrac{1}{2}B_k B_k(u_{j,j}u_i - u_{i,j}u_j)]N_i\,\mathrm{dS} \\ &- \int_{G^-} T_{jk} u_{i,k} u_{i,j}\,\mathrm{dV} \Bigg\} \left\{ \frac{1}{1+\nu} \int_{G^-} \left(\frac{\nu}{1-2\nu} e_{kk}e_{ll} + e_{kl}e_{kl} \right) \mathrm{dV} \right\}^{-1}, \end{aligned} \tag{1.6}$$

while the constraints here are (cf. [1], (7.12), (7.15))

$$\begin{aligned} &B_i = e_{ijk}A_{kj} \text{ (or div } \mathbf{B} = 0), \quad e_{ijk}B_{k,j} = 0 \text{ (or curl } \mathbf{B} = \mathbf{0}), \quad \mathbf{x} \in G^+; \\ &\mathbf{A} = \text{constant (or } (\mathbf{B}, \mathbf{N}) = 0), \quad \mathbf{x} \in \partial G; \\ &\mathbf{B} \to \mathbf{c}(\mathbf{x}), \quad |\mathbf{x}| \to \infty, \end{aligned} \tag{1.7}$$

and

$$T_{ij,j} = 0, \quad \mathbf{x} \in G^-; \quad T_{ij}N_j = -\tfrac{1}{2}(\mathbf{B}, \mathbf{B})N_i, \quad \mathbf{x} \in \partial G, \tag{1.8}$$

and for the perturbed potential ψ,

$$\Delta\psi = 0, \quad \mathbf{x} \in G^+; \qquad \frac{\partial\psi}{\partial N} = (B_j u_{i,j} - B_{i,j} u_j)N_i, \quad \mathbf{x} \in \partial G;$$
$$\psi \to 0, \quad |\mathbf{x}| \to \infty. \tag{1.9}$$

In the next section, the above results will be further elaborated for the special case of an infinitely long beam, which is periodically supported. In Section 3, explicit buckling values are calculated for one beam of arbitrary cross-section. The third section also serves as a first acquaintance with the mathematical methods that will be used in Section 4 to solve the buckling problem for a set of two parallel rods. Buckling values are calculated for both ferromagnetic and superconducting rods. In the final section we present some special results and we compare our results with those following from a more simplified approach, based upon a generalization of the law of Biot and Savart.

2. The slender beam

Consider an infinitely long beam of arbitrary cross-section. The beam is periodically supported (simply supported or clamped), the distance between the supports being l. Let R be a characteristic length for the cross-section. Then, the beam is called slender if $R/l \ll 1$. A coordinate system $\{O\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is chosen with the $\mathbf{e}_3$-axis along the central line of the beam, and the $\mathbf{e}_1$- and $\mathbf{e}_2$-axes in the plane of the cross-section D^- along the principle axes of inertia. It is assumed that in buckling the beam deflects in the $\mathbf{e}_1$-direction. We denote the deflection of the central line of the beam in the $\mathbf{e}_1$-direction by $w(z)$. In accordance with Bernoulli's theory for the bending of slender beams, we then choose the displacement field in an arbitrary point (x, y, z) of the beam as

$$u_1 = w(z) + \tfrac{1}{2}\nu(x^2 - y^2)w''(z),$$
$$u_2 = \nu xyw''(z), \quad u_3 = -xw'(z). \tag{2.1}$$

where ν is Poisson's ratio and $' = \mathrm{d}/\mathrm{d}z$.

The results recapitulated in Section 1 have been derived in [1] under the restriction that the body is of finite dimension. In the above example, however, this is no longer true. We can avoid this discrepancy by assuming that the fields are periodic in the z- or $\mathbf{e}_3$-direction with period p (p is related to l, but does not need to be equal to l, and depends on the type of support). In this case it is allowed to replace in (1.1) and (1.6) the finite region G^- with boundary ∂G by the finite parts of one period. Considering a final cross-section, separating two periods, we notice that the contributions due to points just before and just after this cross-section cancel each other. Hence, the truncated part of ∂G only consists of the lateral surface of the beam. Therefore, from now on one must read for G^- the finite domain of one period, say $z \in (0, p)$, and for ∂G the lateral surface of G^-.

We now are able to evaluate the integral in the denominator of the right-hand sides of (1.1) and (1.6). Since the integral represents the elastic energy of the beam, it is not surprising to find that (2.1) implies that this term is equal to the classical energy for a slender beam in bending (apart from a factor $E/2$), i.e.

$$\frac{1}{1+\nu}\int_{G^-}\left(\frac{\nu}{1-2\nu}\,e_{kk}e_{ll} + e_{kl}e_{kl}\right)\mathrm{dV} = I_y\int_0^p w''^2(z)\,\mathrm{dz}, \tag{2.2}$$

where

$$I_y = \int_{D^-} x^2\,\mathrm{dS}, \tag{2.3}$$

the moment of inertia about the y-axis. Note that in the derivation of (2.2) it is used that

$$\| R^n w^{(n)}(z) \| = \mathrm{O}((R/l)^n)\,\| w(z) \|, \tag{2.4}$$

and that $\mathrm{O}(R^2/l^2)$-terms are neglected with respect to unity.

We assume that the bias field $\mathbf{B}_0$ for the ferromagnetic beam is perpendicular to the $\mathbf{e}_3$-axis, and that for the superconducting beam the unperturbed current runs in the $\mathbf{e}_3$-direction. For both cases, the problem for the rigid-body field is then purely two-dimensional, i.e. $\mathbf{B} = \mathbf{B}(x, y)$ and $(\mathbf{B}, \mathbf{e}_3) = 0$. The problem for the perturbed potential ψ can be reduced to a two-dimensional problem by the separation of variables

$$\psi(x, y, z) - \psi_0 = \phi(x, y)w(z), \text{ (F)}, \tag{2.5}$$

$$\psi(x, y, z) = \phi(x, y)w(z), \text{ (S)}.$$

Note: We try, whenever possible, to treat the ferromagnetic and the superconducting case simultaneously. However when distinction is necessary, we label the ferromagnetic relations with a suffix **(F)** and the superconducting ones with **(S)**.

The separation according to (2.5) is only then consistent with the constraint $\Delta\psi = 0$ if $w(z)$ satisfies the relation

$$w''(z) + \lambda^2 w(z) = 0, \tag{2.6}$$

where the real parameter λ is a separation constant, which is related to l through the support conditions (e.g., for a cantilever $\lambda = \pi/2l$, and for a simply supported beam $\lambda = \pi/l$). The parameter λ is proportional to and always of the same order as l^{-1} and, hence, the parameter δ defined by

$$\delta = \lambda R \; (= \mathrm{O}(R/l) \ll 1), \tag{2.7}$$

is very small. Note that δ is a measure for the slenderness of the beam. With (2.5) and (2.6), the constraint $\Delta\psi = 0$, for $\mathbf{x} \in G^+$, transforms into the following constraint for ϕ

$$\Delta\phi(x, y) = \lambda^2\phi(x, y), \quad (x, y) \in D^+, \tag{2.8}$$

where, now, Δ is the two-dimensional Laplace operator and D^+ is the domain of the vacuum in the $\mathbf{e}_1$, $\mathbf{e}_2$-plane.

3. Buckling values for a single ferromagnetic or superconducting beam

Consider a slender beam as described in the preceding section. For the ferromagnetic case the beam is supposed to be placed in a uniform magnetic field $\mathbf{B}_0$. The basic field $\mathbf{B}_0$ is directed in the $\mathbf{e}_1$-direction, which is taken as the axis of lowest bending stiffness. The deflection in buckling is then indeed in the $\mathbf{e}_1$-direction. This also holds true for the superconducting beam. At this stage we can eliminate a small inconvenience in our formulation. The normalized fields $\hat{\mathbf{B}}$, $\hat{T}$ and $\hat{\psi}$ (see [1], (7.17)) are not dimensionless. Therefore, we introduce new dimensionless normalized variables by (R is a characteristic measure for the cross-section)

$$\hat{\hat{\mathbf{B}}} = \frac{2\pi R}{\mu_0 I_0}\mathbf{B}, \quad \hat{\hat{\mathbf{A}}} = \frac{2\pi R}{\mu_0 I_0}\mathbf{A}, \quad \hat{\hat{\psi}} = \frac{2\pi R}{\mu_0 I_0}\psi, \quad \hat{\hat{T}} = \frac{(2\pi R)^2}{\mu_0 I_0^2}T, \text{ (S)}. \tag{3.1}$$

This normalization also implies that the normalized pre-stresses $\hat{\hat{T}}_{ij}$ are of the same order of magnitude with respect to the small parameter δ as the magnetic components $\hat{\hat{B}}_i$. There are only two minor changes due to this modification. Firstly, in the left-hand side of (1.6) we must replace

$$\frac{E}{\mu_0 I_0^2} \rightarrow \frac{(2\pi R)^2 E}{\mu_0 I_0^2} \text{ (S)}, \tag{3.2}$$

and secondly, with the current I_0 in the $\mathbf{e}_3$-direction, the constraint at infinity (1.7)[4] can be made explicit, yielding (see also [1], (7.5)) (omitting the hats from now on)

$$\mathbf{B} = \frac{R}{|\mathbf{x}|}(-\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y), \quad |\mathbf{x}| \rightarrow \infty, \text{ (S)}, \tag{3.3}$$

where $|\mathbf{x}| = (x^2 + y^2)^{1/2}$ and θ is the pole angle.

We proceed with an evaluation of the numerators of the right-hand sides of (1.1) and (1.6). The magnetic vector potential $\mathbf{A}$ for the rigid-body problem is of the form $\mathbf{A} = \mathbf{A}(x, y)\mathbf{e}_3$, yielding $\mathbf{B} = \mathbf{B}(x, y)$ and $(\mathbf{B}, \mathbf{e}_3) = B_3 = 0$. Moreover, we assume that the supports of the beams are such that the stresses in the $\mathbf{e}_3$- or z-direction are zero, i.e.

$$T_{xz} = T_{yz} = T_{zz} = 0. \tag{3.4}$$

Together with the above results, we use the constraint relations for **B** and ψ of Section 1, and the equations (2.1), (2.2), (2.5) and (2.6) of Section 2. Finally, for the sake of simplicity, we neglect in the superconducting case (S) lateral contraction (i.e. $\nu = 0$). This results in the following asymptotic relations for the buckling values, deduced from (1.1) and (1.6),

$$\frac{\mu_0 EI_y \lambda^4}{B_0^2} = \int_{\partial D} B_x \frac{\partial}{\partial N}(\phi + B_x)\,ds + O(\delta^2),\ \delta \to 0,\ (\mathbf{F}), \tag{3.5.1}$$

$$\frac{4\pi^2 R^2 EI_y \lambda^4}{\mu_0 I_0^2} = \int_{\partial D}\left[-(\phi + B_x)\frac{\partial B_x}{\partial N} - \frac{\delta^2 x}{2R^2}(B_x^2 + B_y^2)N_x\right] ds + O(\delta^4),\ \delta \to 0,\ (\mathbf{S}). \tag{3.5.2}$$

We note that, due to $(1.4)^4$, the (irrelevant) constant ψ_0 does not contribute to (3.5.1) (in fact, the condition for ψ at infinity implies $\psi_0 = 0$). Moreover, we make the convention that any term in the subsequent analysis of the form $O(\delta^n \log^k \delta)$ will be referred to as an $O(\delta^n)$-term.

For a complete solution we still need **B** and ϕ. The intermediate field **B** can be solved from (1.2) or (1.7), whereas the perturbed potential $\phi = \phi(x, y)$ has to satisfy

$$\begin{aligned} &\Delta\phi = \lambda^2\phi,\quad (x, y) \in D^+; \qquad \phi \to 0,\quad x^2 + y^2 \to \infty; \\ &\phi + B_x = 0,\ (\mathbf{F}),\quad \frac{\partial}{\partial N}(\phi + B_x) = 0,\quad (\mathbf{S}),\ (x, y) \in \partial D. \end{aligned} \tag{3.6}$$

It will turn out (see (3.12)) that the leading terms in the right-hand sides of (3.5) are of $O(1)$ with respect to δ, for (F), and $O(\delta^2)$, for (S), which means that the higher-order terms in (3.5) are indeed negligible.

Let the region D^-, occupied by the cross-section of the beam in the x-y-plane, be finite, simply connected and sufficiently regular (in order that all of the manipulations that follow are allowed). Furthermore, let z be the normalized complex variable

$$z = (x + iy)/R, \tag{3.7}$$

and S^-, S^+ and C the regions in the z-plane corresponding with D^-, D^+ and ∂D, respectively. Then, there exists exactly one conformal mapping

$$z = h(u) \tag{3.8}$$

from the region $\{u \mid |u| < 1\}$ in the complex u-plane onto S^+, such that

$$h(-1) = -A,\quad h(1) = B,\quad h(\infty) = \infty,$$

where $-A$ and B are the intersections of the boundary C of S^+ with the negative and positive real axis in the z-plane, respectively (see Fig. 1). For this conformal mapping the number c defined by

$$c = \lim_{u\to\infty} |h(u)/u| = \lim_{u\to\infty} |h'(u)|, \tag{3.9}$$

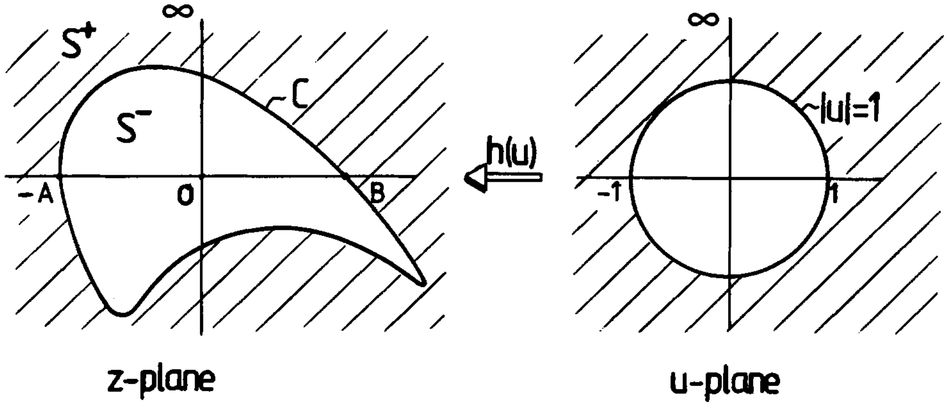

Fig. 1. The conformal mapping $z = h(u)$.

is finite and positive. Moreover, it is assumed that the cross-section is sufficiently regular providing that

$$c = O(1) \quad \text{and} \quad 1/c = O(1), \quad \delta \to 0, \tag{3.10}$$

implying that

$$\lambda Rc = \delta c = O(R/l) \ll 1. \tag{3.11}$$

We can now give the final results for the buckling values. We shall present these first, together with some interpretations and specific results for special cross-sections, and we shall postpone the proof until Subsection 3.1 at the end of this section.

The final formulae for the buckling values, which follow from (3.9) are

$$\frac{2\pi}{\mu_0 E I_y \lambda^4} B_0^2 = \Gamma(\delta c)\{1 + O(\delta^2)\}, \text{ (F)}, \tag{3.12.1}$$

$$\frac{\mu_0}{4\pi E I_y} I_0^2 = \frac{\lambda^2}{\Gamma(\delta c) - \frac{1}{2}}\{1 + O(\delta^2)\}, \text{ (S)}, \tag{3.12.2}$$

for $\delta \to 0$, where

$$\Gamma(\delta c) = -\gamma - \log(\tfrac{1}{2}\delta c), \quad \gamma = 0.577 \quad \text{(Euler's constant)}. \tag{3.13}$$

Before proving these results (in Subsection 3.1), we make some remarks. Firstly, the general form of the results (3.12) holds irrespective of the shape of the cross-section. In fact, the shape of the cross-section only enters these formulae through the number c. Hence, realizing that I_y is proportional to R^4, we see that, apart from a logarithmic factor, the buckling values B_0 and I_0 are proportional to $(R/l)^2$ and to R/l, respectively, for every finite cross-section.

Secondly, for a circular cross-section one has $c = 1$, and then the obtained results correspond completely with those known in literature (cf. [2], [3] for (F), and [4] for (S)). For a beam of elliptic cross-section ($a \times b$; $a \leqslant b$) one gets

$$Rc = \tfrac{1}{2}(a + b), \quad I_y = \tfrac{1}{4}\pi a^3 b. \tag{3.14}$$

Restricting ourselves to case (F) for a cantilever ($\lambda = \pi/2l$), we find from (3.12.1) for the buckling field

$$\frac{B_0^2}{\mu_0 E} = \frac{b}{8a}\left(\frac{\pi a}{2l}\right)^4 \Gamma\left(\frac{\pi(a+b)}{4l}\right). \tag{3.15}$$

This result is in correspondence with [3], eq. (6.13), with $\mu^{-1} \to 0$.

Finally we consider a ferromagnetic cantilever of rectangular cross-section ($a \times b$; $a \leqslant b$). For a rectangle it can be proved that c becomes (analogously to [5], p. 178)

$$Rc = \frac{a}{2[E(p^2) - (1 - p^2)K(p^2)]}, \tag{3.16}$$

where $p \in (0, \frac{1}{2}\sqrt{2})$ is the root of the relation

$$\frac{a}{b} = \frac{E(p^2) - (1 - p^2)K(p^2)}{E(1 - p^2) - p^2 K(1 - p^2)}, \quad \left(0 < \frac{a}{b} \leqslant 1\right), \tag{3.17}$$

and K and E are complete elliptic integrals of the first and second kind, respectively. Moreover

$$I_y = \tfrac{4}{3}a^3 b. \tag{3.18}$$

Then, (3.12.1) yields (with $\lambda = \pi/2l$)

$$\frac{B_0^2}{\mu_0 E} = \frac{2b}{3a\pi}\left(\frac{\pi a}{2l}\right)^4 \Gamma\left(\frac{\pi Rc}{2l}\right). \tag{3.19}$$

In a previous paper [6], one of the authors stated that it was to be expected that the buckling values for a narrow rectangular cross-section may be approximated by the corresponding values for an elliptic cross-section. To check this statement, we shall compare the result (3.19) with (3.15) for an ellipse ($a_1 \times b_1$), such that

$$a_1 = \sigma a, \quad b_1 = \sigma b, \quad \sigma = \frac{2}{(3\pi)^{1/4}}. \tag{3.20}$$

In this case the rectangle and the ellipse have identical thickness-to-width ratio's and moments of inertia I_y. Defining q as the quotient of the buckling values we then find from (3.15) and (3.19)

$$q\left(\frac{a}{b}\right) = \frac{(B_0)_{rectangle}}{(B_0)_{ellipse}} = \left(\frac{\Gamma(\pi Rc/2l)}{\Gamma(\pi(a_1 + b_1)/4l)}\right)^{1/2}. \tag{3.21}$$

Some q-values, for $b/l = 0.1$ and varying a/b are listed in Table 1. These values justify the expectation stated above.

Table 1. Ratio of buckling values for rectangular and elliptic cross-sections

a/b	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Rc/b	1.180	1.121	1.061	1.000	0.938	0.875	0.810	0.743	0.672	0.595
q	0.982	0.983	0.984	0.985	0.987	0.990	0.994	1.000	1.008	1.022

We now proceed with the proof of the general results (3.12).

3.1. *Proof of (3.12)*

All manipulations in this section will be performed in the complex z-plane (with z according to (3.7)). We shall not give detailed references in all steps of our calculations, but for a general reference with respect to the methods we use here we refer to [7]. Introducing the complex line element dz by

$$R\,\mathrm{d}z = i(N_x + iN_y)\,\mathrm{d}S = iN\,\mathrm{d}s, \tag{3.22}$$

where (N_x, N_y) denotes the unit outward normal on C, and the complex derivative as

$$\frac{\partial}{\partial z} = \tfrac{1}{2}R\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{3.23}$$

we immediately derive the useful relation

$$2\,\mathrm{d}z\frac{\partial}{\partial z} = \mathrm{d}s\left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial N}\right), \quad z \in C. \tag{3.24}$$

With the function F defined as

$$F = B_x - iB_y, \quad z \in S^+ \cup C, \tag{3.25}$$

we rewrite the constraints (1.2) and (1.7), (3.3) for the intermediate state as

$$\begin{aligned} &F \text{ analytical}, && z \in S^+,\\ &iF\,\mathrm{d}z \in \mathbb{R},\ (\mathbf{F});\qquad F\,\mathrm{d}z \in \mathbb{R},\ (\mathbf{S}), && z \in C,\\ &F = 1 + \mathrm{O}(z^{-2}),\ (\mathbf{F});\qquad F = -iz^{-1} + \mathrm{O}(z^{-2}),\ (\mathbf{S}), && z \to \infty. \end{aligned} \tag{3.26}$$

For the perturbed potential ϕ we have at our disposal the constraints (3.6). Consider ϕ as $\phi(z, \bar{z})$, then the Helmholtz equation (3.6)[1] can be written as ($\delta = \lambda R$)

$$\Delta_z\phi = 4\frac{\partial^2\phi(z, \bar{z})}{\partial z\partial\bar{z}} = \delta^2\phi(z, \bar{z}). \tag{3.27}$$

With the introduction of the real-valued function

$$f = f(z, \bar{z}) := \phi + B_x, \quad z \in S^+ \cup C, \tag{3.28}$$

the constraints (3.6)[2,3] simplify to

$$f = 0, \text{(F)}, \quad \frac{\partial f}{\partial N} = 0, \text{(S)}, \quad z \in C. \tag{3.29}$$

After substitution of (3.25) and (3.28) into (3.5) and with the use of (3.24) and the above constraints, the buckling formulae (3.5) can be transformed into

$$\frac{\mu_0 EI_y \lambda^4}{2B_0^2} = \operatorname{Im} \int_C F \frac{\partial f}{\partial z} \mathrm{d}z, \text{(F)}, \tag{3.30.1}$$

$$\frac{4\pi^2 EI_y \lambda^2 \delta^2}{\mu_0 I_0^2} = \operatorname{Im} \int_C \left[-f \frac{\mathrm{d}F}{\mathrm{d}z} + \tfrac{1}{4}\delta^2 (z + \bar{z}) F^2 \right] \mathrm{d}z, \text{(S)}. \tag{3.30.2}$$

In the above equations $\partial f/\partial z$ and f occur. Therefore, we first derive integral equations for $\partial f/\partial z$ (F) and f (S) on C.

The fundamental solutions of the Laplace and the Helmholtz equation are

$$G(z, \bar{z}, z_0, \bar{z}_0) = -\frac{1}{2\pi} \log |z - z_0| = -\frac{1}{4\pi} [\log (z - z_0) + \log (\bar{z} - \bar{z}_0)],$$

$$H(z, \bar{z}, z_0, \bar{z}_0) = \frac{1}{2\pi} K_0(\delta |z - z_0|), \tag{3.31}$$

respectively, where K_0 is the modified Bessel function of the second kind of order zero. These solutions satisfy

$$\Delta_z G = -\delta_D(z - z_0); \qquad \Delta_z H - \delta^2 H = -\delta_D(z - z_0), \tag{3.32}$$

where $\delta_D(z)$ is Dirac's delta function. Green's second identity, together with (3.26) and (3.27), implies for $(x_0, y_0) \in D^+$,

$$\phi(x_0, y_0) = \int_{\partial D} \left(\phi \frac{\partial H}{\partial N} - H \frac{\partial \phi}{\partial N} \right) \mathrm{d}s, \tag{3.33}$$

and

$$B_x(x_0, y_0) = B_x(\infty) + \int_{\partial D} \left(B_x \frac{\partial G}{\partial N} - G \frac{\partial B_x}{\partial N} \right) \mathrm{d}s$$

$$= B_x(\infty) + \int_{\partial D} \left(B_x \frac{\partial H}{\partial N} - H \frac{\partial B_x}{\partial N} \right) \mathrm{ds}$$

$$+ \int_{\partial D} \left(B_y \frac{\partial (H - G)}{\partial s} - B_x \frac{\partial (H - G)}{\partial N} \right) \mathrm{ds}, \tag{3.34}$$

where in the latter step it is used that $\partial B_x / \partial N = -\partial B_y / \partial s$ on ∂D and one partial integration is performed.

With use of (3.24) it can be shown that

$$\phi \frac{\partial H}{\partial N} \mathrm{ds} = \mathrm{Re} \left\{ \frac{2}{i} \phi \frac{\partial H}{\partial z} \mathrm{dz} \right\}. \tag{3.35}$$

Analogously, and with the use of (3.25), the last integral of (3.34) can be transformed into a complex integral. Then (3.33) and (3.34) add up to

$$f(z_0, \bar{z}_0) = \mathrm{Re} \left\{ F(\infty) + \frac{2}{i} \int_C f \frac{\partial H}{\partial z} \mathrm{dz} - \frac{2}{i} \int_C H \frac{\partial f}{\partial z} \mathrm{dz} \right.$$

$$\left. - \frac{2}{i} \int_C F \frac{\partial (H - G)}{\partial z} \mathrm{dz} \right\}, \quad z_0 \in S^+. \tag{3.36}$$

We note that both $\partial H / \partial z$ and $\partial G / \partial z$ are of the form

$$- \frac{1}{4\pi (z - z_0)} + \text{regular term},$$

for $z \to z_0$. Therefore, for $z_0 \to C$ only the first integral on the right-hand side of (3.36) becomes singular. Using Plemelj's formulae (cf. [7], or see (3.57)) we then obtain from (3.36) by letting $z_0 \to C$ (since z_0 and $\bar{z}_0$ are coupled on C, we denote $f(z_0, \bar{z}_0)$ by $f(z_0)$ for z_0 on C)

$$f(z_0) = 2\mathrm{Re} \left\{ F(\infty) + \frac{2}{i} ⨍_C f \frac{\partial H}{\partial z} \mathrm{dz} - \frac{2}{i} \int_C H \frac{\partial f}{\partial z} \mathrm{dz} \right.$$

$$\left. - \frac{2}{i} \int_C F \frac{\partial (H - G)}{\partial z} \mathrm{dz} \right\}, \quad z_0 \in C, \tag{3.37}$$

where ⨍ stands for Cauchy's principal value (cf. [7]). At this point we have to consider for a moment the ferromagnetic and the superconducting case separately. From the relations (3.29) (with the first one written as $\partial f / \partial s = 0$) it follows that

$$i \frac{\partial f}{\partial z} \mathrm{dz} \in \mathbb{R}, \ (\mathbf{F}); \qquad \frac{\partial f}{\partial z} \mathrm{dz} \in \mathbb{R}, \ (\mathbf{S}), \quad z \in C. \tag{3.38}$$

Using these relations and the last rule of (3.26) in (3.37) we deduce successively

$$\frac{2}{i}\int_C H\frac{\partial f}{\partial z}\,\mathrm{d}z = 1 - \mathrm{Re}\left\{\frac{2}{i}\int_C F\frac{\partial(H-G)}{\partial z}\,\mathrm{d}z\right\}, \ (\mathbf{F}), \tag{3.39.1}$$

and

$$f(z_0) = \mathrm{Re}\left\{\frac{4}{i}\int_C f\frac{\partial H}{\partial z}\,\mathrm{d}z - \frac{4}{i}\int_C F\frac{\partial(H-G)}{\partial z}\,\mathrm{d}z\right\}, \ (\mathbf{S}), \tag{3.39.2}$$

both for $z_0 \in C$.

In order to get a better uniformity between the (**F**) and (**S**) case, we introduce the auxiliary function $\Lambda(s) \in \mathbb{R}$ by

$$\Lambda(s) := \frac{1}{i}\int_0^s \frac{\partial f}{\partial z}(z(s))\frac{\mathrm{d}z(s)}{\mathrm{d}s}\,\mathrm{d}s - \frac{\kappa_F}{L}s, \quad 0 \leqslant s \leqslant L, \tag{3.40}$$

where s is the arc length parameter along C which stands in a one-to-one relationship with z on C, i.e. $z = z(s)$ on C. Moreover, L is the total arc length of C and

$$\kappa_F := \frac{1}{i}\int_C \frac{\partial f}{\partial z}\,\mathrm{d}z \in \mathbb{R}. \tag{3.41}$$

From (3.40) it follows that

$$\frac{1}{i}\frac{\partial f}{\partial z}\,\mathrm{d}z = \left[\frac{\mathrm{d}\Lambda(s)}{\mathrm{d}s} + \frac{\kappa_F}{L}\right]\mathrm{d}s, \quad z \in C, \quad 0 \leqslant s \leqslant L. \tag{3.42}$$

After the substitution of (3.42) into (3.39.1) and one partial integration, (3.39.1) transforms into

$$-2\int_C \Lambda\frac{\partial H}{\partial s}\,\mathrm{d}s = -4\,\mathrm{Re}\int_C \Lambda\frac{\partial H}{\partial z}\,\mathrm{d}z = 1 - \frac{2\kappa_F}{L}\int_C H\,\mathrm{d}s$$
$$- \mathrm{Re}\left\{\frac{2}{i}\int_C F\frac{\partial(H-G)}{\partial z}\,\mathrm{d}z\right\}, \ (\mathbf{F}), \quad z_0 \in C. \tag{3.43}$$

The integral equations (3.39.2) and (3.43) are too complicated to solve them exactly. However, recalling that δ is very small, we can write

$$2\pi(H - G) = \Gamma(\delta) + O(\delta^2),$$
$$4\pi\frac{\partial(H-G)}{\partial z} = \tfrac{1}{2}\delta^2\{\Gamma(\delta) + \tfrac{1}{2} - \log|z - z_0|\}(\bar{z} - \bar{z}_0) + O(\delta^4), \tag{3.44}$$

for $\delta \to 0$, uniform in $z, z_0 \in C$. Introducing the first order approximations for $i\Lambda(z)$ and $f(z)$ by ($ig_F \in \mathbb{R}$ and $g_S \in \mathbb{R}$)

$$i\Lambda(z) = g_F(z)(1 + O(\delta^2)), \text{(F)}; \quad f(z) = \delta^2 g_S(z)(1 + O(\delta^2)), \text{(S)}, \tag{3.45}$$

respectively, and neglecting terms of $O(\delta^2)$ with respect to unity, we can approximate (3.43) and (3.39.2) by

$$\mathrm{Re}\left\{\frac{1}{2\pi i}\int_C \frac{g_F(z)}{z - z_0}\,\mathrm{d}z\right\} = \tfrac{1}{2} + \frac{\kappa_f}{2\pi L}\int_C \log|z - z_0|\,\mathrm{d}s - \frac{\kappa_F}{2\pi}\Gamma(\delta), \text{(F)}, \tag{3.46.1}$$

and

$$\tfrac{1}{2}g_S(z_0) + \mathrm{Re}\left\{\frac{1}{2\pi i}\int_C \frac{g_S(z)}{z - z_0}\,\mathrm{d}z\right\}$$

$$= \mathrm{Re}\left\{\frac{1}{4\pi i}\int_C F(z)[\log|z - z_0| - \Gamma(\delta) - \tfrac{1}{2}](\bar{z} - \bar{z}_0)\,\mathrm{d}z\right\}, \text{(S)}, \tag{3.46.2}$$

for $z_0 \in C$. Use of the above definitions and approximations in (3.30) results in the following set of buckling relations

$$\frac{\mu_0 EI_y \lambda^4}{2B_0^2} = \mathrm{Im}\left\{\int_C F\frac{\mathrm{d}g_F}{\mathrm{d}s}\,\mathrm{d}s + \frac{i\kappa_F}{L}\int_C F\,\mathrm{d}s\right\}, \text{(F)}, \tag{3.47.1}$$

and

$$\frac{4\pi^2 EI_y \lambda^2}{\mu_0 I_0^2} = \mathrm{Im}\left\{\int_C F\frac{\mathrm{d}g_S}{\mathrm{d}s}\,\mathrm{d}s + \frac{1}{4}\int_C (z + \bar{z})F^2\,\mathrm{d}z\right\}, \text{(S)}. \tag{3.47.2}$$

Hence, for the calculation of the buckling values we do not need to know g_F and g_S completely, but we only need the values of the first integrals on the right-hand sides of (3.47). Again the calculation of these integrals runs for (F) and (S) mainly along the same lines. Firstly, we define two real-valued functions $R_F(z_0, \bar{z}_0)$ and $R_S(z_0, \bar{z}_0)$, for $z_0 \in S^- \cup C$, which for $z_0 \in C$ are equal to the right-hand sides of (3.46), i.e.

$$R_F(z_0, \bar{z}_0) = \tfrac{1}{2} + \frac{\kappa_F}{2\pi L}\int_C \log|z - z_0|\,\mathrm{d}s - \frac{\kappa_F}{2\pi}\Gamma(\delta), \tag{3.48.1}$$

and

$$R_S(z_0, \bar{z}_0) = \mathrm{Re}\left\{\frac{1}{4\pi i}\int_C F(z)[\log|z - z_0| - \Gamma(\delta) - \tfrac{1}{2}](\bar{z} - \bar{z}_0)\,\mathrm{d}z\right\}, \tag{3.48.2}$$

for $z_0 \in S^- \cup C$.

For later use we calculate the first derivatives with respect to z_0 of these functions for $z_0 \in S^-$. They read (use (3.26)^{3,5})

$$\frac{\partial R_F}{\partial z_0} = -\frac{\kappa_F}{4\pi L}\int_C \frac{\mathrm{d}s}{z - z_0}, \tag{3.49.1}$$

and

$$\frac{\partial R_S}{\partial z_0} = \frac{1}{16\pi i}\int_C F(z)\left[\log|z_0 - z_0|^2 - \frac{\bar{z} - \bar{z}_0}{z - z_0}\right]\mathrm{d}z + \tfrac{1}{4}i\Gamma(\delta), \quad z_0 \in S^-. \tag{3.49.2}$$

The real and continuous function $R(z_0, \bar{z}_0)$ possesses continuous derivatives in S^- and, furthermore, it can be proved that

$$\frac{\partial^2 R}{\partial z_0 \partial \bar{z}_0} = 0, \quad z_0 \in S^-. \tag{3.50}$$

For R_F the proof of (3.50) is trivial (see (3.49.1)), whereas, for R_S, (3.50) follows from (3.49.2) with the use of the property that for $z_0 \in S^-$

$$\frac{1}{2\pi i}\int_C \frac{F(z)}{z - z_0}\,\mathrm{d}z = F(\infty) = 0. \tag{3.51}$$

We note that, due to (3.50) and because $F\,\mathrm{d}z \in \mathbb{R}$ on C, the integral

$$\frac{1}{4\pi}\int_C F(z)\log|z - z_0|^2\,\mathrm{d}z, \quad z_0 \in S^-,$$

occurring in the right-hand side of (3.49.2) is a real constant which will be denoted by κ_S, i.e., (take $z_0 = 0$)

$$\kappa_S = \frac{1}{2\pi}\int_C F(z)\log|z|\,\mathrm{d}z. \tag{3.52}$$

The relation (3.50) together with the properties of $R(z_0, \bar{z}_0)$ mentioned above imply the existence of analytical functions $\Psi_F(z_0)$ and $\Psi_S(z_0)$ for $z_0 \in S^- \cup C$, such that

$$R(z_0, \bar{z}_0) = \operatorname{Re}\Psi(z_0), \quad z_0 \in S^- \cup C. \tag{3.53}$$

Differentiating (3.53) with respect to z_0, we obtain a relation, which will be used further on,

$$\frac{\mathrm{d}\Psi(z_0)}{\mathrm{d}z_0} = 2\frac{\partial R(z_0, \bar{z}_0)}{\partial z_0}, \quad z_0 \in S^-. \tag{3.54}$$

As a second step, we introduce the Cauchy integral

$$\Phi(z_0) = \frac{1}{2\pi i}\int_C \frac{g(z)}{z - z_0}\,\mathrm{d}z, \quad z_0 \in \mathbb{C}\backslash C. \tag{3.55}$$

Then

(i) $\Phi(z)$ analytical, $z \in \mathbb{C}\backslash C$;

$$\text{(ii)}\ \Phi(z) = O(z^{-1}), \quad z \to \infty; \tag{3.56}$$

$$\text{(iii)}\ \Phi^{\pm}(z_0) = \mp\tfrac{1}{2}g(z_0) + \frac{1}{2\pi i}\int_C \frac{g(z)}{z - z_0}\,\mathrm{d}z, \quad z_0 \in C. \tag{3.57}$$

The relations (iii) are the well-known Plemelj-formulae, already mentioned before. Combining these relations with the integral equations (3.46) and using that ig_F, g_S and the right-hand sides of (3.46), i.e., R_F and R_S, are all real, we straightforwardly find that

$$\mathrm{Re}\,\Phi^-(z_0) = R(z_0), \quad z_0 \in C. \tag{3.58}$$

Furthermore, for case (F), it follows that

$$\mathrm{Re}\,\Phi_F^+(z_0) = \mathrm{Re}\,\Phi_F^-(z_0), \quad z_0 \in C. \tag{3.59}$$

Finally, subtraction of the Plemelj-formulae amounts to

$$\frac{\mathrm{d}g}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}(\Phi^- - \Phi^+), \text{ along } C. \tag{3.60}$$

A comparison of (3.58) with (3.53) yields

$$\mathrm{Re}(\Phi^-(z_0) - \Psi(z_0)) = 0, \quad z_0 \in C. \tag{3.61}$$

Use of a well-known result from the theory of complex functions, saying that if the real part of an analytical function is zero at a boundary C, this function can at most be an imaginary constant in the interior region S^- of C, now implies that

$$\Phi^-(z_0) = \Psi(z_0) + i\mu, \quad z_0 \in S^- \cup C, \tag{3.62}$$

where μ is an irrelevant real constant.

With the preceding results we can derive

$$\int_C F\frac{\mathrm{d}g}{\mathrm{d}s}\,\mathrm{d}s = \int_C F\frac{\mathrm{d}}{\mathrm{d}s}(\Phi^- - \Phi^+)\,\mathrm{d}s = \int_C F\frac{\mathrm{d}\Psi}{\mathrm{d}z}\,\mathrm{d}z, \tag{3.63}$$

since

$$\int_C F\left(\frac{\mathrm{d}\Phi}{\mathrm{d}z}\right)^+ \mathrm{d}z = 0, \tag{3.64}$$

because $F\,\mathrm{d}\Phi/\mathrm{d}z$ tends to zero as (at least) $O(z^{-2})$ at infinity. Explicit expressions for $\mathrm{d}\Psi/\mathrm{d}z$ can be deduced from (3.54) and (3.49). From (3.54) and (3.49.1) we obtain

$$\frac{\mathrm{d}\Psi_F}{\mathrm{d}z_0} = -\frac{\kappa_F}{2\pi L}\left(\int_C \frac{\mathrm{d}s}{z - z_0}\right)^-$$

$$= -\frac{\kappa_F}{2\pi L}\left(\int_C \frac{\mathrm{d}s}{z - z_0}\right)^+ - i\frac{\kappa_F}{L}\frac{\mathrm{d}s}{\mathrm{d}z}(z_0), \quad z_0 \in C, \tag{3.65.1}$$

with the use of the Plemelj formulae. Analogously, we obtain from (3.54) and (3.49.2) together with (3.51) and (3.52)

$$\frac{\mathrm{d}\Psi_S}{\mathrm{d}z_0} = -\frac{1}{8\pi i}\left(\int_C \frac{\bar{z}F(z)}{z - z_0}\mathrm{d}z\right)^- + \tfrac{1}{2}i\left[\Gamma(\delta) - \kappa_S\right]$$

$$= -\frac{1}{8\pi i}\left(\int_C \frac{\bar{z}F(z)}{z - z_0}\mathrm{d}z\right)^+ - \tfrac{1}{4}\bar{z}_0 F(z_0) + \tfrac{1}{2}i\left[\Gamma(\delta) - \kappa_S\right], \quad z_0 \in C. \tag{3.65.2}$$

We now have to substitute (3.65) into (3.63) and, subsequently, the result into (3.47). After some elementary calculations we finally arrive at the following buckling relations

$$\frac{\mu_0 E I_y \lambda^4}{2B_0^2} = \kappa_F(1 + O(\delta^2)), \text{ (F)}, \tag{3.66.1}$$

$$\frac{4\pi E I_y \lambda^2}{\mu_0 I_0^2} = (\Gamma(\delta) - \kappa_S - \tfrac{1}{2})(1 + O(\delta^2)), \text{ (S)}. \tag{3.66.2}$$

At this point we still have to determine the constants κ_F and κ_S. It is only in this last step that the conformal mapping (3.8) (and, hence, the specific shape of the cross-section) enters in our analysis. For the calculation of κ_F we do not use its definition (3.4.1), but a result that is a consequence of (3.58)–(3.59). Considering $\Phi_F^+(z) = \Phi_F^+(h(u)) = \tilde{\Phi}_F^+(u)$, $|u| \geqslant 1$, as a function of u, we see that

$$\frac{1}{2\pi i}\int_{|u|=1} \mathrm{Re}\,\tilde{\Phi}_F^+ \frac{\mathrm{d}u}{u} = \mathrm{Re}\left\{\frac{1}{2\pi}\int_{|u|=1} \tilde{\Phi}_F^+ \frac{\mathrm{d}u}{iu}\right\} = \mathrm{Re}\{\Phi_F^+(\infty)\} = 0. \tag{3.67}$$

On the other hand (3.59), (3.58) and (3.48.1) imply

$$\frac{1}{2\pi i}\int_{|u|=1} \mathrm{Re}\Phi^+ \frac{du}{u} = \frac{1}{2\pi i}\int_{|u|=1} R_F \frac{du}{u}$$

$$= \frac{1}{2}\left(1 - \frac{\kappa_F}{\pi}\Gamma(\delta)\right)\frac{1}{2\pi i}\int_{|u|=1}\frac{du}{u} + \frac{\kappa_F}{2\pi L}\frac{1}{2\pi i}\int_{|u|=1}\frac{du}{u}\int_C \log|z - z_0|\, ds_0 \tag{3.68}$$

$$= \frac{1}{2}\left(1 - \frac{\kappa_F}{\pi}\Gamma(\delta)\right) + \frac{\kappa_F}{2\pi L}\int_C I(u_0, \bar{u}_0)\, ds_0,$$

where

$$I(u_0, \bar{u}_0) = \frac{1}{2\pi i}\int_{|u|=1} \log|h(u) - h(u_0)|\frac{du}{u} \ (\in \mathbb{R}). \tag{3.69}$$

Extending the domain of I to $|u_0| \geqslant 1$, we see that I satisfies

$$(i)\ \frac{\partial I}{\partial u_0} = -\frac{1}{2}\frac{h'(u_0)}{2\pi i}\int_{|u|=1}\frac{du}{u(h(u) - h(u_0))} = \frac{1}{2u_0}, \quad |u_0| > 1, \tag{3.70}$$

$$(ii)\quad I = \log|cu_0| + O(u_0^{-1}), \quad |u_0| \to \infty; \tag{3.71}$$

in accordance with (3.9). Therefore, the real integral I is equal to

$$I(u_0, \bar{u}_0) = \log|cu_0| = \log c + \log|u_0|, \quad |u_0| \geqslant 1. \tag{3.72}$$

Substitution of (3.72) into (3.68) with simultaneous use of (3.67) leads us to

$$1 - \frac{\kappa_F}{\pi}\Gamma(\delta) + \frac{\kappa_F}{\pi}\log c = 0, \tag{3.73}$$

or, with the definition (3.13),

$$\kappa_F = \frac{\pi}{\Gamma(\delta) - \log c} = \frac{\pi}{\Gamma(\delta c)}. \tag{3.74}$$

With $z = h(u)$ the expression (3.52) for κ_S becomes

$$\kappa_S = \frac{1}{2\pi}\int_C F(z)\log|h(u)|\, dz$$

$$= \frac{1}{2\pi}\int_C F(z)\left[\log c + \log\left|\frac{h(u)}{cu}\right| - \log|u|\right] dz$$

$$= \frac{\log c}{2\pi}\int_C F(z)\, dz + \mathrm{Re}\frac{1}{2\pi}\int_C F(z)\log\left(\frac{h(u)}{\hat{c}u}\right) dz, \tag{3.75}$$

because of the (S)-property that $F\,dz \in \mathbb{R}$, for $z \in C$. Here, $\hat{c}$ is a complex constant such that $h(u)/\hat{c}u \to 1$, for $u \to \infty$ (hence, $|\hat{c}| = c$). Since $h(u)/\hat{c}u$ is an analytical function, unequal to zero, in S^+, the function $\log(h(u)/\hat{c}u)$ is also analytical in S^+ and tends to zero for $u \to \infty$. Moreover $F(z) = -iz^{-1} + O(z^{-2})$, for $z \to \infty$ and, hence the second integral in the last right-hand side of (3.75) is zero, while the first one becomes equal to 2π. Hence, (3.75) amounts to

$$\kappa_S = \log c. \tag{3.76}$$

Substitution of (3.74) and (3.76) into the buckling relations (3.66) ultimately results in

$$\frac{\mu_0 E I_y \lambda^4}{2B_0^2} = \frac{\pi}{\Gamma(\delta c)}(1 + O(\delta^2)), \text{ (F)}, \tag{3.77.1}$$

$$\frac{4\pi E I_y \lambda^2}{\mu_0 I_0^2} = (\Gamma(\delta c) - \tfrac{1}{2})(1 + O(\delta^2)), \text{ (S)}. \tag{3.77.2}$$

This completes the proof of (3.12).

4. A set of two parallel beams

In this section we consider systems of two identical, parallel, infinitely long beams (as described in Section 2). The beams can be either soft ferromagnetic (F) or superconducting (S). In order to keep our analysis manageable, we restrict ourselves to cross-sections which show double symmetry. The distance between the centres of the cross-sections is $2a$. A coordinate system $\{O\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is chosen with the origin O midway between the centres of the cross-sections, the $\mathbf{e}_3$-axis parallel to the central lines of the beams and the $\mathbf{e}_1$-axis through the centres of the cross-sections. The $\mathbf{e}_1$-axis coincides with one of the symmetry axes of the cross-sections. In the $\mathbf{e}_1\mathbf{e}_2$-plane the cross-sections are denoted by D_1 and D_2, with boundaries ∂D_1 and ∂D_2, respectively, and the vacuum region is denoted by D^+. The centre of D_1 lies on the positive $\mathbf{e}_1$-axis (coordinates $(a, 0)$). Our general approach applies to arbitrary, however doubly symmetric, cross-sections, but explicit numerical results will only be given for circular cross-sections.

The basic field B_0 (case (F)) is taken in the $\mathbf{e}_1$-direction (in Section 5 we shall discuss the somewhat more complicated case that $\mathbf{B}_0$ makes an arbitrary angle θ_0 with the $\mathbf{e}_1$-axis). In case (S), we assume that the total currents in the two superconducting beams are equal in magnitude (I_0), but these currents can be either in equal (case (S_e)) or in opposite directions (case (S_o)) (the current in the first beam, cross-section D_1, is always in the positive $\mathbf{e}_3$-direction). Then, in all these cases the buckling displacement is along the $\mathbf{e}_1$-axis. Moreover, we assume that the buckling displacements of the two beams are equal but opposite. In fact, if the beams buckle in the same direction, the complete set behaves as one beam and it turns out that then the buckling load is much higher (i.e., at least $O(\delta^{-1})$; see the preceding section) than the one we shall find in this section. Thus, we suppose that the displacement field (2.1) (for one beam) here generalizes into

$$
\begin{aligned}
u_1(x, y) &= w(z) + \tfrac{1}{2}\nu[(x-a)^2 - y^2]\,w''(z),\\
u_2(x, y) &= \nu(x-a)\,y w''(z), \qquad\qquad (4.1)\\
u_3(x, y) &= -(x-a)\,w'(z), \quad (x, y) \in D_1^-,
\end{aligned}
$$

yielding the same expression for the elastic energy of one beam as found in (2.2). Obviously, the elastic energies of the two beams are equal.

Again, all manipulations in this section will be performed in the complex z-plane, with z according to (3.7) and with all notations as introduced in Section 3 (e.g., $\partial D_1 \to C_1$). There is no difficulty in verifying that, in analogy with Section 3, the relations (3.26) remain valid here. We only have to replace the explicit condition at infinity for case (S) by the more vague condition

$$F \to 0, \quad z \to \infty, \text{ (S)}. \qquad (4.2)$$

This vagueness is due to the fact that the behaviour of F for $z \to \infty$ is different in the cases $(\mathrm{S_e})$ and $(\mathrm{S_o})$; in case $(\mathrm{S_e})$ one has $F = \mathrm{O}(z^{-1})$, $z \to \infty$, whereas $F = \mathrm{O}(z^{-2})$, $z \to \infty$, in case $(\mathrm{S_o})$. Of great use in the following calculations are the symmetry relations (which are due to the double symmetry of the cross-sections)

$$
\begin{aligned}
&F(-z) = F(z), \text{ (F)},\\
&F(-z) = -F(z), (\mathrm{S_e}); \qquad F(-z) = F(z), (\mathrm{S_o}).
\end{aligned}
\qquad (4.3)
$$

Finally, we need the following results for the integrals of $F(z)$ along C_1, which can easily be verified,

$$\int_{C_1} F\,\mathrm{d}z = 0, \text{ (F)}; \qquad \int_{C_1} F\,\mathrm{d}z = 2\pi, \text{ (S)}. \qquad (4.4)$$

Likewise, the relations for the perturbed potential ϕ remain practically the same as in Section 3. We only have to realize that (for $w > 0$) the displacement of D_1 is in the positive $\mathbf{e}_1$-direction, but that of D_2 is in the *negative* $\mathbf{e}_1$-direction. Therefore, instead of $f = \phi + B_x$ as in Section 3, we must here introduce

$$f = f(z, \bar{z}) = \begin{cases} \phi + B_x, & \operatorname{Re} z > 0,\\ \phi - B_x, & \operatorname{Re} z < 0. \end{cases} \qquad (4.5)$$

However, we do not have to worry about this somewhat peculiar relationship, if we make use of the trivial symmetry relations

$$
\begin{aligned}
&f(-z, -\bar{z}) = -f(z, \bar{z}), \text{ (F)};\\
&f(-z, -\bar{z}) = f(z, \bar{z}), (\mathrm{S_e}); \qquad f(-z, -\bar{z}) = -f(z, \bar{z}), (\mathrm{S_o}).
\end{aligned}
\qquad (4.6)
$$

Thus, the relations (3.6) for ϕ remain valid ($\partial D \to \partial D_1$) and the same holds true for the relations (3.29) for $f(z)$, which now refer to C_1. Finally the formulae (3.30) for the buckling value may be applied here too, where the integration takes place over C_1. However, in the (S)-formulae the $O(\delta^2)$-term in the integral on the right-hand side of (3.30.2) may now be neglected, as the leading term in this case turns out to be of $O(1)$. Since the influence of the pre-stresses is enclosed in this $O(\delta^2)$-term, this means that the pre-stresses may be neglected now (note that this was not the case for the single superconducting beam).

Considering (3.30) we conclude that we are only interested in the functions $F(z)$ and $f(z)$, for $z \in C_1$. In the same way as in the preceding section we can derive the following integral equation for $f(z)$, $z \in C_1$ (compare with the derivation of (3.37) from (3.33)–(3.36), and realize that now $C = C_1 \cup C_2$ and $f = \phi - B_x$ on C_2)

$$f(z_0) = 2\,\mathrm{Re}\left\{F(\infty) + \frac{2}{i}\fint_C f\frac{\partial H}{\partial z}\,\mathrm{d}z - \frac{2}{i}\int_C H\frac{\partial f}{\partial z}\,\mathrm{d}z - \frac{2}{i}\int_C F\frac{\partial(H-G)}{\partial z}\,\mathrm{d}z + \frac{4}{i}\int_{C_2} F\frac{\partial H}{\partial z}\,\mathrm{d}z\right\}, \quad z_0 \in C_1. \tag{4.7}$$

The last term in the right-hand side of (4.7) is descended from

$$2\int_{C_2}\left(B_x\frac{\partial H}{\partial N} - H\frac{\partial B_x}{\partial N}\right)\mathrm{d}s = \mathrm{Re}\left\{\frac{4}{i}\int_{C_2} F\frac{\partial H}{\partial z}\,\mathrm{d}z\right\}, \quad z_0 \in C_1, \tag{4.8}$$

in the derivation of which a.o. the relation $\partial B_x/\partial N = -\partial B_y/\partial s$ is used.

In exact analogy with the preceding section we introduce, in the **(F)**-case, the auxiliary function $\Lambda(s)$ by (3.40) and we denote the first-order approximation of $i\Lambda$ by $g_F(z)$ (see (3.45.1)). Moreover, we introduce the first-order approximation of $f(z)$ in the (S)-case according to

$$f(z) = g_S(z)(1 + O(\delta^2)), \tag{4.9}$$

(note that this is in contrast with $(3.45)^2$). Using the approximations (3.44), the boundary conditions (3.29), the conditions at infinity $(3.26)^4$ and (4.2), and neglecting all terms of order $O(\delta^2)$ we derive from (4.7) the following two integral equations for $g_F(z)$ and $g_S(z)$ ($ig_F \in \mathbb{R}$, $g_S \in \mathbb{R}$)

$$\mathrm{Re}\left\{\frac{1}{2\pi i}\fint_C \frac{g_F(z)}{z - z_0}\,\mathrm{d}z\right\} = R_F(z_0), \quad z_0 \in C_1, \tag{4.10.1}$$

and

$$\tfrac{1}{2}g_S(z_0) + \mathrm{Re}\left\{\frac{1}{2\pi i}\fint_C \frac{g_S(z)}{z - z_0}\,\mathrm{d}z\right\} = R_S(z_0), \quad z_0 \in C_1, \tag{4.10.2}$$

where

$$R_F(z_0) = \frac{\kappa_F^{(1)}}{2\pi L}\int_{C_1} \log|z - z_0|\,\mathrm{d}s + \frac{\kappa_F^{(2)}}{2\pi L}\int_{C_2} \log|z - z_0|\,\mathrm{d}s$$

$$- \frac{\Gamma(\delta)}{2\pi}(\kappa_F^{(1)} + \kappa_F^{(2)}) + \tfrac{1}{2} - \mathrm{Re}\left\{\frac{1}{2\pi i}\int_{C_2}\frac{F(z)}{z - z_0}\,\mathrm{d}z\right\}, \quad z_0 \in C_1, \tag{4.11}$$

and

$$R_S(z_0) = \mathrm{Re}\left\{\frac{1}{\pi i}\int_{C_2}\frac{F(z)}{z - z_0}\,\mathrm{d}z\right\}, \quad z_0 \in C_1. \tag{4.12}$$

Furthermore

$$\kappa_F^{(1)} = \frac{1}{i}\int_{C_1}\frac{\partial f}{\partial z}\,\mathrm{d}z, \quad \kappa_F^{(2)} = \frac{1}{i}\int_{C_2}\frac{\partial f}{\partial z}\,\mathrm{d}z; \qquad L = \int_{C_1}\mathrm{d}s. \tag{4.13}$$

From the symmetry relations (4.6) it is evident that

$$\kappa_F^{(2)} = -\kappa_F^{(1)}. \tag{4.14}$$

Hence, in the expression (4.11) the third term vanishes and the first two terms can be taken together to yield

$$R_F(z_0) = \mathrm{Re}\left\{\frac{\kappa_F^{(1)}}{2\pi L}\int_{C_1}\log\left(\frac{z - z_0}{z + z_0}\right)\mathrm{d}s\right\} + \tfrac{1}{2} - \mathrm{Re}\left\{\frac{1}{2\pi i}\int_{C_2}\frac{F(z)}{z - z_0}\,\mathrm{d}z\right\}, \quad z_0 \in C_1. \tag{4.15}$$

Note that the integral equations (4.10) still contain integrals over C_2. However, these integrals can with the use of the symmetry relations for $F(z)$ and $f(z)$ easily be transformed in integrals over C_1 (but this is postponed for the moment).

The integral equations (4.10) are similar to the equations (3.46) of Section 3, and just as in that section these equations will be solved by Hilbert-methods. The analysis is exactly the same as the one presented in Section 3 between the eqs. (3.53)–(3.63) and, therefore, we immediately give the results, which read

$$\frac{\mu_0 E I_y \lambda^4}{2B_0^2} = \mathrm{Im}\int_{C_1} F\frac{\partial f}{\partial z}\,\mathrm{d}z = \mathrm{Im}\left\{\int_{C_1} F\left(\frac{\mathrm{d}g_F}{\mathrm{d}s} + \frac{i\kappa_F^{(1)}}{L}\right)\mathrm{d}s\right\}$$

$$= \mathrm{Im}\left\{\frac{i\kappa_F^{(1)}}{L}\int_{C_1} F\,\mathrm{d}s + \int_{C_1} F\left(\frac{\mathrm{d}\Psi_F}{\mathrm{d}z} - \frac{\mathrm{d}\Phi_F^{+}}{\mathrm{d}z}\right)\mathrm{d}z\right\}, \text{ (F)}, \tag{4.16.1}$$

and

$$\frac{4\pi^2 EI_y \lambda^2 \delta^2}{\mu_0 I_0^2} = -\operatorname{Im} \int_{C_1} \frac{dF}{dz} g_S \, dz = \operatorname{Im} \int_{C_1} F \frac{dg_S}{ds} \, ds$$
$$= \operatorname{Im} \left\{ \int_{C_1} F \left(\frac{d\Psi_S}{dz} - \frac{d\Phi_S^+}{dz} \right) dz \right\}, \text{ (S)}. \tag{4.16.2}$$

Note that here (3.64) does not apply, because the region exterior to C_1 is not simply connected but contains as a hole the region S_2, corresponding to the cross-section of the second beam.

The functions $\Psi(z)$ and $\Phi^+(z)$, occurring in (4.16), must be calculated from (compare with (3.57)–(3.58))

$$\operatorname{Re} \Phi_F^+(z_0) = \operatorname{Re} \Phi_F^-(z_0) = R_F(z_0) = \operatorname{Re} \Psi_F(z_0), \quad z_0 \in C_1, \text{ (F)}; \tag{4.17.1}$$

$$\operatorname{Im} \Phi_S^+(z_0) = \operatorname{Im} \Phi_S^-(z_0), \quad \operatorname{Re} \Phi_S^-(z_0) = R_S(z_0) = \operatorname{Re} \Psi_S(z_0), \quad z_0 \in C_1, \text{ (S)}; \tag{4.17.2}$$

Up to here the results apply to arbitrary, but doubly symmetric cross-sections. For the explicit calculations of the right-hand sides of (4.16), however, we from now on restrict ourselves to circular cross-sections (radius R, $I_y = \pi R^4/4$). The analysis is based on a conformal mapping from the exterior region S^+ onto a ring. For other than circular cross-sections the use of a conformal mapping is in principle also possible, but in that case our considerations are much more complex.

For two circular cross-sections the conformal mapping reads (in the z-plane all distances are normalized with respect to R)

$$z = \beta \frac{1+u}{1-u}, \quad \beta = \sqrt{m^2 - 1}, \quad m = \frac{a}{R} > 1. \tag{4.18}$$

Under this mapping, the exterior region S^+ transforms into a ring bounded by concentric circles of radii α and α^{-1} where

$$\alpha = m - \beta, \quad \alpha^{-1} = m + \beta, \quad \alpha \in (0, 1), \tag{4.19}$$

(see Fig. 2.)

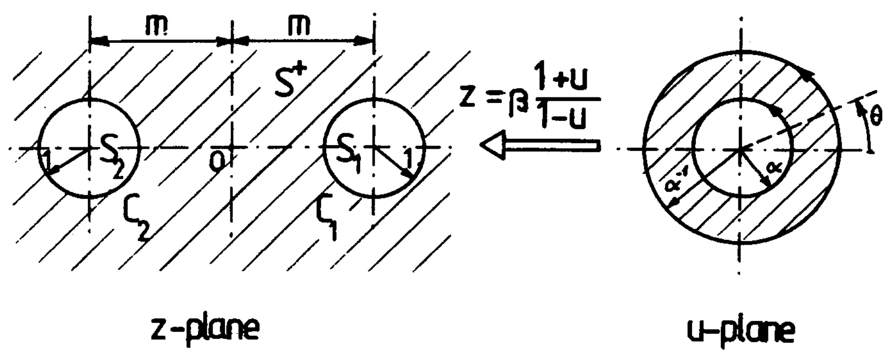

Fig. 2. The conformal mapping (4.18).

The cross-sections S_1 and S_2 are mapped onto the interior and exterior regions of the ring, respectively, and the boundaries C_1 and C_2 onto the circles $|u| = \alpha$ and $|u| = \alpha^{-1}$, respectively. The point $z = \infty$ corresponds to $u = 1$. Finally for $z \in C_1$ (i.e. $|u| = \alpha$ or $u = \alpha e^{i\theta}$) one has

$$dz = \frac{2\beta}{(1-u)^2}\,du = \frac{2i\beta u}{(1-u)^2}\,d\theta. \tag{4.20}$$

From here on the paths for the two cases (**F**) and (**S**) diverge, and, therefore, we have to consider these cases separately. We start with

The ferromagnetic case (**F**)

For the calculation of $F(z)$, it is convenient to introduce the function

$$G(u) = \frac{u(\tilde{F}(u) - 1)}{(1-u)^2}, \quad \tilde{F}(u) = F(z(u)), \quad \alpha \leqslant |u| \leqslant \alpha^{-1}. \tag{4.21}$$

From $(3.26)^4$ it follows that

$$\tilde{F}(u) = 1 + O((1-u)^2), \quad u \to 1, \tag{4.22}$$

and, hence, $G(u)$ is regular for $u = 1$. From $(3.26)^{1,2}$, (4.3) and (4.4), we conclude that

$$\begin{aligned}
&(i)\ G(u) \ \text{analytical}, \qquad \alpha < |u| < \alpha^{-1},\\
&(ii)\ G(u) + \frac{u}{(1-u)^2} \in \mathbb{R}, \quad |u| = \alpha,\\
&(iii)\ G(u) = G(u^{-1}), \quad \alpha < |u| < \alpha^{-1},\\
&(iv)\ \int_{|u|=\alpha} \frac{G(u)}{u}\,du = 0.
\end{aligned} \tag{4.23}$$

Developing $G(u)$ in a Laurent series and employing the properties of $G(u)$ listed in (4.23) we conclude that $G(u)$ must be of the form

$$G(u) = \sum_{n=1}^{\infty} g_n(u^n + u^{-n}); \qquad g_n = \frac{n\alpha^{2n}}{1-\alpha^{2n}}, \quad n \geqslant 1. \tag{4.24}$$

This yields for F

$$\tilde{F}(u) = F_0 + \sum_{n=1}^{\infty} F_n(u^n + u^{-n}), \tag{4.25}$$

with

$$F_0 = 1 + 2g_1, \quad F_n = g_{n+1} - 2g_n + g_{n-1}, \quad n \geqslant 1, \quad g_0 = 0. \tag{4.26}$$

For the solution of (4.17.1) we need explicit expressions for the integrals occurring in the right-hand side of (4.15). Using the symmetry of F (i.e. (4.3)[1]) and (4.20) we deduce with (4.25) that, for $z_0 \in S_1 \cup C_1$,

$$-\frac{1}{2\pi i}\int_{C_2}\frac{F(z)}{z - z_0}\,dz = -\frac{1}{2\pi i}\int_{C_1}\frac{F(z)}{z + z_0}\,dz$$

$$= -\frac{1}{2\pi i}\int_{|u|=\alpha}\tilde{F}(u)\left(\frac{1}{1-u} + \frac{1}{u - u_0^{-1}}\right)du$$

$$= \sum_{n=1}^{\infty} F_n\,(u_0^n - 1) = g_1 + \sum_{n=1}^{\infty} F_n u_0^n, \quad |u_0| \leqslant \alpha. \tag{4.27}$$

The first integral in the right-hand side of (4.15) is calculated by transforming the path of integration C into the circle $|u| = \alpha$, developing $\log(1 - v)$, ($v = uu_0^{-1}$ or $v = uu_0$) in a power series in v and applying Cauchy's residue theorem. In this way we obtain ($L = 2\pi$)

$$\operatorname{Re}\left\{\frac{\kappa_F^{(1)}}{2\pi L}\int_{C_1}\log\left(\frac{z - z_0}{z + z_0}\right)ds\right\}$$

$$= \frac{\kappa_F^{(1)}}{2\pi}\operatorname{Re}\left\{\frac{1}{2\pi i}\int_{|u|=\alpha}[\log(-u_0) + \log(1 - uu_0^{-1}) - \log(1 - uu_0)]\right.$$

$$\left.\times\left[\frac{1}{u - \alpha^2} - \frac{1}{u - 1}\right]du\right\}$$

$$= \frac{\kappa_F^{(1)}}{2\pi}\left\{\log\alpha - \operatorname{Re}\sum_{n=1}^{\infty}\frac{(1 - \alpha^{2n})}{n}u_0^n\right\}, \quad |u_0| = \alpha. \tag{4.28}$$

The right-hand side $R_F(z_0)$ of (4.17.1) is now explicitly known. The symmetry relation for $f(z, \bar{z})$ implies ($\tilde{\Phi}(u) = \Phi(z(u))$)

$$\Phi_F^+(-z) = -\Phi_F^+(z), \quad \tilde{\Phi}_F^+(u^{-1}) = -\tilde{\Phi}_F^+(u). \tag{4.29}$$

Furthermore, the function Ψ_F must be analytical in the inner region $|u| < \alpha$. Consequently, the Laurent series for Φ_F and Ψ_F are of the form

$$\tilde{\Phi}_F^+(u) = \sum_{n=1}^{\infty}\phi_n(u^n - u^{-n}), \quad \alpha \leqslant |u| \leqslant \alpha^{-1}, \tag{4.30.1}$$

and

$$\tilde{\Psi}_F(u) = \sum_{n=0}^{\infty}\psi_n u^n, \qquad |u| \leqslant \alpha. \tag{4.30.2}$$

Substituting (4.27) and (4.28) and the series (4.30) into (4.17.1), we see that these integral equations are satisfied if

$$\frac{\kappa_F^{(1)}}{2\pi} = -\frac{1+2g_1}{\log\alpha^2} = -\frac{1}{\log\alpha^2}\left(\frac{1+\alpha^2}{1-\alpha^2}\right),$$

$$\psi_n = F_n - \frac{\kappa_F^{(1)}}{2\pi}\frac{1-\alpha^{2n}}{n}, \quad n \geqslant 1, \tag{4.31}$$

$$\psi_0 = 0, \quad \phi_n = -\frac{\alpha^{2n}}{1-\alpha^{2n}}\psi_n, \quad n \geqslant 1.$$

It is now a matter of simple algebra to derive that for $z \in C_1$ or $|u| = \alpha$,

$$\begin{aligned}\left(\frac{\mathrm{d}\Phi_F^+}{\mathrm{d}z} - \frac{\mathrm{d}\Psi_F}{\mathrm{d}z}\right)\mathrm{d}z &= \frac{\mathrm{d}}{\mathrm{d}u}(\tilde{\Phi}_F^+ - \tilde{\Psi}_F)\,\mathrm{d}u \\ &= \sum_{n=1}^{\infty}\left(\frac{\kappa_F^{(1)}}{2\pi} - \frac{nF_n}{1-\alpha^{2n}}\right)(u^n + \alpha^{2n}u^{-n})\frac{\mathrm{d}u}{u}.\end{aligned} \tag{4.32}$$

The final step consists of the substitution of (4.32) into the buckling formula (4.16.1) and the calculation of the thus obtained integrals. This leads to the following explicit result for the buckling field ($I_y = \pi R^4/4$)

$$\begin{aligned}\frac{\mu_0 E\delta^4}{16B_0^2} &= \mathrm{Im}\left\{\frac{i\kappa_F^{(1)}}{2\pi}\frac{1}{2\pi i}\int_{C_1}\frac{F(z)}{z-m}\mathrm{d}z\right. \\ &\quad \left. - \sum_{n=1}^{\infty}\left(\frac{\kappa_F^{(1)}}{2\pi} - \frac{nF_n}{1-\alpha^{2n}}\right)\cdot\frac{1}{2\pi i}\int_{|u|=\alpha}\tilde{F}(u)(u^n + \alpha^{2n}u^{-n})\frac{i\,\mathrm{d}u}{u}\right\} \\ &= \frac{\kappa_F^{(1)}}{2\pi}\left(1 + g_1 + \sum_{n=1}^{\infty}F_n\alpha^{2n}\right) + \sum_{n=1}^{\infty}F_n(1+\alpha^{2n})\left(\frac{nF_n}{1-\alpha^{2n}} - \frac{\kappa_F^{(1)}}{2\pi}\right) \\ &= -\frac{1}{\log\alpha^2}\left(\frac{1+\alpha^2}{1-\alpha^2}\right)^2 + \sum_{n=1}^{\infty}nF_n^2\left(\frac{1+\alpha^{2n}}{1-\alpha^{2n}}\right) > 0, \ (\mathbf{F}).\end{aligned} \tag{4.33}$$

In the above calculation we have used that

$$\frac{1}{2\pi i}\int_{C_1}\frac{F(z)}{z-m}\mathrm{d}z = \frac{1}{2\pi i}\int_{C_1\cup C_2}\frac{F(z)}{z-m}\mathrm{d}z - \frac{1}{2\pi i}\int_{C_2}\frac{F(z)}{z-m}\mathrm{d}z, \tag{4.34}$$

where the first integral in (4.34) is equal to $F(\infty) = 1$ and the second is given by (4.27) for $u_0 = \alpha^2$. Moreover, in the final step we have used that

$$\sum_{n=1}^{\infty}F_n = -g_1, \quad 1 + 2g_1 = \frac{1+\alpha^2}{1-\alpha^2}. \tag{4.35}$$

The coefficients F_n follow, for given α, from (4.26) and (4.24). According to its definition (4.18)–(4.19),

$$\alpha = m - \beta = \frac{a}{R} - \sqrt{\frac{a^2}{R^2} - 1}, \tag{4.36}$$

the number α is directly related to the ratio a/R and, hence, our final result (4.33) represents an explicit expression for the buckling field as a function of the ratio a/R. Numerical results will be presented in the final section of this paper.

The superconducting case (S)

In this case we introduce $G(u)$ as

$$G(u) = i\beta u \frac{\tilde{F}(u)}{(1-u)^2}, \quad \alpha \leqslant |u| \leqslant \alpha^{-1}, \tag{4.37}$$

and this function satisfies (note that $\tilde{F}(1) = 0$, see (3.26))

$$\begin{aligned} &(i)\ G(u) \text{ analytical}, && \alpha < |u| < \alpha^{-1}, u \neq 1, \\ &(ii)\ G(u) \in \mathbb{R}, && |u| = \alpha, \\ &(iii)\ G(u) = \mathrm{O}((1-u)^{-1}), \quad u \to 1, \\ &(iv)\ \frac{1}{2\pi i}\int_{|u|=\alpha} \frac{G(u)}{u}\,\mathrm{d}u = \tfrac{1}{2}, \end{aligned} \tag{4.38}$$

while the symmetry condition here reads (cf. (4.3))

$$G(u) = -G(u^{-1}),\ (\mathrm{S_e}); \qquad G(u) = G(u^{-1}),\ (\mathrm{S_o}). \tag{4.39}$$

The properties (4.38) together with the symmetry condition (4.39) yield

$$G(u) = \frac{1}{1-u} - \sum_{n=-\infty}^{\infty} g_n u^n; \qquad g_n = -g_{-n} = \frac{\alpha^{2n}}{1+\alpha^{2n}}, \quad n \geqslant 1, \quad g_0 = \tfrac{1}{2},\ (\mathrm{S_e}), \tag{4.40.1}$$

and

$$G(u) = \tfrac{1}{2},\ (\mathrm{S_o}). \tag{4.40.2}$$

We can now determine F from (4.37) and next the integral in the right-hand side of (4.12). As before, Ψ_S is taken equal to this integral and, thus,

$$\Psi_S(z_0) = -\frac{1}{\pi i}\int_{C_2}\frac{F(z)}{z - z_0}\,\mathrm{d}z = \frac{1}{\pi i}\int_{C_1}\frac{F(z)}{z + z_0}\,\mathrm{d}z$$

$$= \frac{2i}{\beta}\left[-\tfrac{1}{2} + g_1 + \sum_{n=1}^{\infty} F_n u_0^n\right],\ F_n = g_{n+1} - 2g_n + g_{n-1},\quad |u_0| \leqslant \alpha,\ (\mathrm{S_e}),$$

$$(4.41.1)$$

$$\Psi_S(z_0) = \frac{2i}{z_0 + \beta} = \frac{i}{\beta}(1 - u_0),\quad |u_0| \leqslant \alpha,\ (\mathrm{S_o}). \qquad (4.41.2)$$

Accounting for the symmetry conditions for Φ_S^+, we write the function $\tilde{\Phi}_S^+(u)$, $\alpha \leqslant |u| \leqslant \alpha^{-1}$, as

$$\tilde{\Phi}_S^+(u) = \sum_{n=-\infty}^{\infty} i\phi_n u^n = \begin{cases} i\sum_{n=0}^{\infty}\phi_n(u^n + u^{-n}),\ (\mathrm{S_e}),\\[2ex] i\sum_{n=1}^{\infty}\phi_n(u^n - u^{-n}),\ (\mathrm{S_o}).\end{cases} \qquad (4.42)$$

The coefficients ϕ_n, $n \geqslant 1$ follow from the relation $\mathrm{Im}\Phi_S^+ = \mathrm{Im}\Psi_S$, for $|u| = \alpha$ (the constant ϕ_0 is irrelevant, but can be chosen such that $\Phi_S^+(z \to \infty) = \tilde{\Phi}_S^+(1) = 0$). This yields

$$\phi_n = \frac{2}{\beta}\frac{\alpha^{2n}}{1 + \alpha^{2n}}F_n,\quad n \geqslant 1,\ (\mathrm{S_e}), \qquad (4.43.1)$$

and

$$\phi_1 = \frac{\alpha}{2\beta^2},\quad \phi_n = 0,\quad n \geqslant 2,\ (\mathrm{S_o}). \qquad (4.43.2)$$

We now are able to evaluate the right-hand side of (4.16.2). Substituting the preceding results and calculating the integrals in the usual way, we finally arrive at (use the Laurent series for F and $I_y = \pi R^4/4$)

$$\frac{\pi^2 E\delta^4 R^2}{\mu_0 I_0^2} = \frac{4}{\beta^2}\sum_{n=1}^{\infty} nF_n^2\left(\frac{1 - \alpha^{2n}}{1 + \alpha^{2n}}\right)$$

$$= \frac{4}{\beta^2}\sum_{n=1}^{\infty}\frac{n\alpha^{4n}(1 - \alpha^{2n})^3}{(1 + \alpha^{2n-2})^2(1 + \alpha^{2n})^3(1 + \alpha^{2n+2})^2} > 0,\ (\mathrm{S_e}), \qquad (4.44)$$

and

$$\frac{\pi^2 E\delta^4 R^2}{\mu_0 I_0^2} = -\frac{1}{2\beta^3}(\alpha + \alpha^{-1}) = -\frac{m}{\beta^3} < 0, \ (S_o). \tag{4.45}$$

Hence, we conclude that in case the currents run in the same direction (S_e) the system buckles in a symmetric mode (i.e., opposite displacements), where the critical current is given by (4.44) as function of the ratio a/R. On the other hand, as the right-hand side of (4.45) is negative, there is no symmetric buckling in the (S_o)-case (opposite currents). This does not imply that the (S_o)-system is always stable, but the critical current is much higher (at least $O(\delta^{-1})$, compare with the case of one beam) than the one for the (S_e)-system.

5. Conclusions and discussion

In this section we look at the results of Section 4 into more detail for some specific cases. Let us first apply the result (4.33) to the case of two cantilevered rods of circular cross-section, radius R, length l. In this case is $\delta = \pi R/2l$ and, then, (4.33) yields

$$\frac{B_0}{\sqrt{\mu_0 E}} = \frac{1}{\sqrt{Q_F}}\left(\frac{\pi R}{4l}\right)^2, \tag{5.1}$$

where $Q_F = Q_F(m)$ stands for the right-hand side of (4.33). This result shows that for fixed $m = a/R$ the buckling load is proportional to R^2/l^2 (just as in the case of one single beam). The dependence of B_0 on the distance between the rods is expressed by the factor Q_F. In Table 2 some values for Q_F as function of m are given.

Table 2. Values for Q_F

m	1.04	1.25	1.43	1.67	2.00	2.50	3.30	5.00	10.0
Q_F	31.7	2.52	1.28	0.788	0.537	0.371	0.277	0.228	0.169

The corresponding $B_0/\sqrt{\mu_0 E}$-values as function of m and for fixed (R/l)-value (i.e. $R/l = 0.01$) are given in Table 3. The data in this table indicate an increase in the buckling value with an increase in the distance between the rods.

Our numerical results are in good correspondence with those of [8] in case $\mu_r = 5.10^4$. In [8] the same problem as mentioned here is treated in a completely different way and for more general values of μ_r (i.e. here μ_r is assumed to be so large that even $\mu_r(\lambda R)^2 \gg 1$, whereas in [8] it is only assumed that $\mu_r \gg 1$ (e.g. $\mu_r > 100$), but $\mu_r(\lambda R)^2$ may remain finite).

A second aspect deserving attention is the influence of the direction of the basic field $\mathbf{B}_0$ with respect to the plane through the two rods. Thus far, we have taken the direction of $\mathbf{B}_0$ parallel to this plane. Let us now consider the more general case that $\mathbf{B}_0$ makes an angle θ_0, $\theta_0 \in [0, \pi/2]$, with the positive $\mathbf{e}_1$-axis. We investigate the influence of the value of θ_0 on the buckling value and we determine the direction of the buckling deflection, which, as we shall show, is not always equal to the direction of $\mathbf{B}_0$.

We assume a symmetrical buckling mode and we denote the angle between the deflection of the first beam and the positive $\mathbf{e}_1$-axis by θ_1. This means that the displacements of the central lines of the first rod are given by

$$u_1(a, 0) = w(z) \cos \theta_1; \qquad u_2(a, 0) = w(z) \sin \theta_1, \tag{5.2}$$

whereas those of the second rod are equal but opposite. For circular cross-sections one has $I_x = I_y = \pi R^4/4$ and, thus, the elastic energy remains as given by (2.2).

Our basic formula (1.1) for the buckling field was derived in [1] by putting a functional J equal to zero (cf. [1], (6.16)–(6.22)). Starting from this formula we here derived a.o. (3.5.1) and (4.16.1). With the displacement field according to (5.2), the functional J depends on θ_1. Analogous to the derivation of (3.5.1) we now obtain

$$J(\theta_1) = \frac{\pi\mu_0 E\delta^4}{4B_0^2} - \int_{\partial D_1} (B_x \cos \theta_1 + B_y \sin \theta_1) \frac{\partial}{\partial N} (\phi + B_x \cos \theta_1 + B_y \sin \theta_1)\, \mathrm{d}s. \tag{5.3}$$

According to [1], the correct value $\hat{\theta}_1$ of θ_1 can be determined by variation of J with respect to θ_1, i.e.,

$$\frac{\mathrm{d}J}{\mathrm{d}\theta_1}(\hat{\theta}_1) \quad \text{and} \quad \frac{\mathrm{d}^2 J}{\mathrm{d}\theta_1^2}(\hat{\theta}_1) > 0. \tag{5.4}$$

The lowest buckling value is then obtained from

$$J(\hat{\theta}_1) = 0. \tag{5.5}$$

The further analysis of this problem runs exactly along the same lines as the one for $\theta_0 = 0$ presented in Section 4. Therefore, we refrain from giving the details of the calculations here. The only extra complication is due to a more general condition at infinity for the analytical function $F(z)$ introduced in (3.25). Instead of (3.26)[4] we must use here

$$F(z) \to e^{-i\theta_0} + O(z^{-2}), \quad |z| \to \infty. \tag{5.6}$$

As in (4.21)–(4.26) we can solve $F(z)$, yielding (compare with (4.24)–(4.26), and note that the F_n's are no longer real)

$$F(z) = F_0 + \sum_{n=1}^{\infty} F_n(u^n + u^{-n}), \tag{5.7}$$

with

$$F_0 = e^{-i\theta_0} + 2g_1, \quad F_n = g_{n+1} - 2g_n + g_{n-1}, \quad n \geqslant 1, \quad g_0 = 0,$$

$$g_n = n\alpha^{2n}\left[\frac{\cos\theta_0}{1 - \alpha^{2n}} + i\frac{\sin\theta_0}{1 + \alpha^{2n}}\right], \quad n \geqslant 1. \tag{5.8}$$

The evaluation of the right-hand side of (5.3) is a generalization of the derivation of (4.16.1) (in which J is already put equal to zero). The result reads (g corresponds to g_F but is no longer an imaginary function; see (3.42) and (3.45))

$$J(\theta_1) = \frac{\pi\mu_0 E\delta^4}{4B_0^2} - \text{Im} \int_{C_1} (\bar{F} + Fe^{2i\theta_1}) \left(\frac{\mathrm{d}g}{\mathrm{d}s} + \frac{i\kappa_F^{(1)}}{2\pi} \right) \mathrm{d}s. \tag{5.9}$$

The function g satisfies (compare with (4.32))

$$\frac{\mathrm{d}g}{\mathrm{d}s}\,\mathrm{d}s = -\sum_{n=1}^{\infty} \left(\frac{\kappa_F^{(1)}}{2\pi} - \frac{nF_n}{1-\alpha^{2n}} \right) (u^n + \alpha^{2n}u^{-n}) \frac{\mathrm{d}u}{u} + \sum_{n=1}^{\infty} \frac{nF_n}{1+\alpha^{2n}} (u^n - \alpha^{2n}u^{-n}) \frac{\mathrm{d}u}{u}, \tag{5.10}$$

while $\kappa_F^{(1)}$ is given by (compare with (4.31)[1])

$$\frac{\kappa_F^{(1)}}{2\pi} = -\frac{e^{-i\theta_0} + 2g_1}{\log \alpha^2}. \tag{5.11}$$

Substituting (5.7), (5.8), (5.10) and (5.11) into (5.9) we obtain

$$\frac{1}{4\pi} J(\theta_1) = \frac{\mu_0 E\delta^4}{16B_0^2} - (c_0 + c_1 \cos 2\theta_1 + c_2 \sin 2\theta_1), \tag{5.12}$$

where the coefficients c_0, c_1 and c_2, which are independent of θ_1, are given by

$$\begin{aligned} c_0 &= \frac{-1}{2\log\alpha^2} |F_0|^2 + \sum_{n=1}^{\infty} \frac{2n\alpha^{2n}}{1-\alpha^{4n}} |F_n|^2 > 0, \\ c_1 &= \frac{-1}{2\log\alpha^2} \text{Re}\,\{F_0^2\} + \sum_{n=1}^{\infty} n \frac{1+\alpha^{4n}}{1-\alpha^{4n}} \text{Re}\,\{F_n^2\}, \\ c_2 &= \frac{1}{2\log\alpha^2} \text{Im}\,\{F_0^2\} - \sum_{n=1}^{\infty} n \frac{1+\alpha^{4n}}{1-\alpha^{4n}} \text{Im}\,\{F_n^2\}. \end{aligned} \tag{5.13}$$

Application of (5.4) to (5.12) yields

$$\tan 2\hat{\theta}_1 = \frac{c_2}{c_1} \quad \text{and} \quad \frac{\cos 2\hat{\theta}_1}{c_1} > 0, \tag{5.14}$$

which after substitution into (5.5) finally results in

$$\frac{\mu_0 E\delta^4}{16B_0^2} = c_0 + (c_1^2 + c_2^2)^{1/2}. \tag{5.15}$$

In Table 3 we have listed some critical B_0-values for various values of m, for $\theta_0 = 0$, $\pi/4$, $\pi/2$ and for $R/l = 0.01$. In this table, $\hat{B}$ represents the bucking value for two rods relative to the value of one rod, which is given by (3.15) in case $a = b = R$. Hence

$$\hat{B} = \frac{B_0^{(2)}}{B_0^{(1)}} = \frac{1}{(2\Gamma(\delta)[c_0 + (c_1^2 + c_2^2)^{1/2}])^{1/2}}. \tag{5.16}$$

Table 3. Relative buckling values for two ferromagnetic rods ($R/l = 10^{-2}$)

m		1.04	1.25	1.43	1.67	2.00	2.50	3.30	5.00	10.00
$\hat{B}$	$\theta_0 = 0$	0.061	0.216	0.302	0.385	0.467	0.547	0.628	0.717	0.833
	$\theta_0 = \frac{1}{4}\pi$	0.086	0.305	0.415	0.477	0.558	0.609	0.665	0.733	0.837
	$\theta_0 = \frac{1}{2}\pi$	0.509	0.541	0.564	0.589	0.617	0.650	0.690	0.747	0.841

From the above table we see that the critical B_0-value depends on the angle of incidence θ_0 of $\mathbf{B}_0$. This is illustrated in the first graph in Fig. 3, which shows a tendency for B_0 to increase when θ_0 increases from 0 to $\pi/2$. In the second graph of Fig. 3 the difference between $\hat{\theta}_1$ and θ_0 is plotted against θ_0. It turns out that $\hat{\theta}_1 = \theta_0$ if $\theta_0 = 0$ or $\theta_0 = \pi/2$. Hence, the deflection and the basic field $\mathbf{B}_0$ are in the same direction when $\mathbf{B}_0$ is either parallel or normal to the plane of the rods. In both cases one has $c_2 = 0$, while in the first case $c_1 > 0$ and in the second case $c_1 < 0$. Furthermore, Fig. 3 shows that the difference between θ_0 and $\hat{\theta}$ is maximal for θ_0 in the neighbourhood of $\pi/4$ and that this difference decreases with increasing m.

As a second example, we shall apply the result (4.44) to the case of two infinitely long superconducting rods of circular cross-section, simply supported over periods of length l. Then, $\delta = \pi R/l$, and (4.44) yields

$$\sqrt{\frac{\mu_0}{E}}\, I_0 = \frac{\pi R}{\sqrt{Q_S}}\left(\frac{\pi R}{l}\right)^2, \tag{5.17}$$

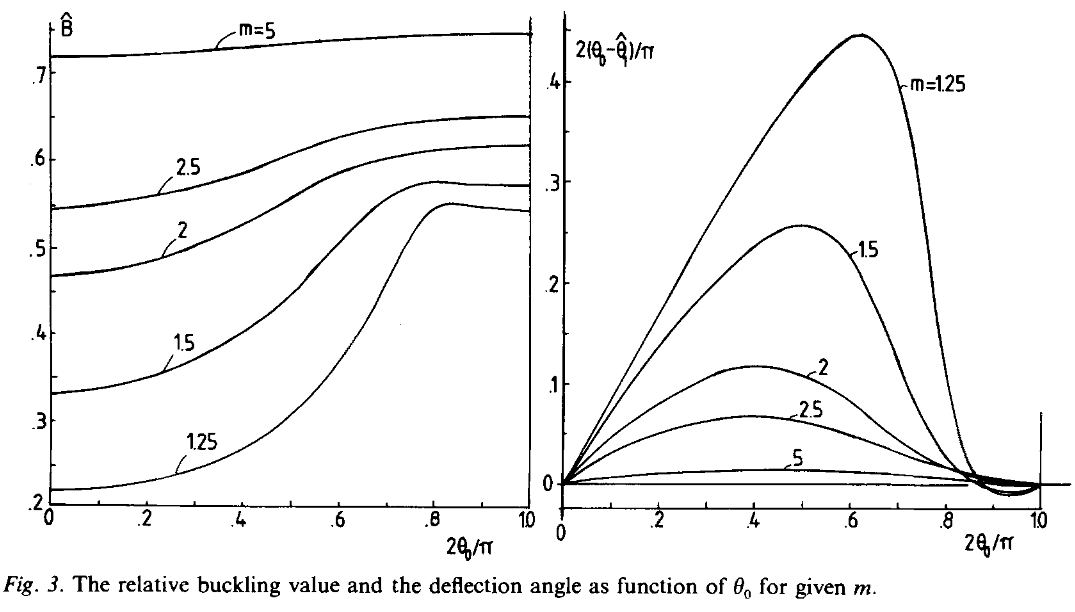

Fig. 3. The relative buckling value and the deflection angle as function of θ_0 for given m.

where $Q_S = Q_S(m)$ stands for the right-hand side of (4.44). This relation formally resembles (5.1), and, hence, the behaviour of I_0 under varying R/l or m is the same as that of B_0. Values of Q_S as function of m are given in Table 4. We note that for larger values of m the factor $1/\sqrt{Q_S}$ approaches m.

Table 4. Values of Q_S

m	1	1.5	2	3	4	6	8	10
Q_S	0.311	0.220	0.168	0.0935	0.0568	0.0266	0.0153	0.00985
$1/\sqrt{Q_S}$	1.79	2.13	2.44	3.27	4.20	6.13	8.09	10.08

It is of some technical interest to compare this result with the result of a less accurate but more simple solution which is based upon a generalization of the law of Biot and Savart. The basic relation for this method is given by Moon in [2], eq. (2-6.4). Let L_1 and L_2 be two curves in $\mathbb{R}_3$ carrying the same electric current I_0. Moreover, let P_1 and P_2 be two points on L_1 and L_2 with position vectors $\mathbf{r}_1(s_1)$ and $\mathbf{r}_2(s_2)$, respectively. Here, s_1 and s_2 are the arc length parameters along L_1 and L_2, respectively. The force per unit of length in P_1 acting on L_1 is now calculated as the Lorentz-force due to the current through L_1 times the magnetic field created by L_2. The latter follows from a generalization of the law of Biot and Savart (cf. [2], (2-6.3)). According to [2], (2-6.4) this force is then given by

$$\mathbf{F}(s_1) = \frac{\mu_0 I_0^2}{4\pi} \int_{L_2} \frac{(\mathbf{t}_1 \times (\mathbf{t}_2 \times \mathbf{R}))}{R^3} \, ds_2, \tag{5.18}$$

where $\mathbf{t}_1$ and $\mathbf{t}_2$ are unit tangent vectors along L_1 and L_2, respectively and $\mathbf{R}$ is the position vector from P_2 to P_1, i.e.

$$\mathbf{t}_1 = \frac{d\mathbf{r}_1}{ds_1}, \quad \mathbf{t}_2 = \frac{d\mathbf{r}_2}{ds_2}, \quad \mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2. \tag{5.19}$$

The above formula for $\mathbf{F}$ is in so far an approximation in that, firstly, the three-dimensional current carrying bodies are considered as one dimensional curves (thus, for instance, the specific shape of the cross-section and the distribution of the current over this cross-section are disregarded) and, secondly, the force due to the self field of L_1 is neglected. Nevertheless, it will turn out that this approach will give good agreement with our results as long as the two current filaments are not too nearby.

We shall now apply the above formula to our problem of two straight, parallel, infinitely long current carriers with equal currents I_0. In the undeformed state, the filaments are a distance $2a$ apart and directed in the $\mathbf{e}_3$-direction. We define $z := s_1$ and $\zeta := s_2$. The deflections of the filaments are directed in the $\mathbf{e}_1$-direction and denoted by $u_1(z)$ and $u_2(\zeta)$, respectively. Hence,

$$\begin{aligned} \mathbf{r}_1 &= [a + u_1(z)]\mathbf{e}_1 + z\mathbf{e}_3, \quad \mathbf{r}_2 = [-a + u_2(\zeta)]\mathbf{e}_1 + \zeta\mathbf{e}_3, \\ \mathbf{R} &= [2a + u_1(z) - u_2(\zeta)]\mathbf{e}_1 + (z - \zeta)\mathbf{e}_3, \\ \mathbf{t}_1 &= u_1'(z)\mathbf{e}_1 + \mathbf{e}_3, \quad \mathbf{t}_2 = u_2'(\zeta)\mathbf{e}_1 + \mathbf{e}_3. \end{aligned} \tag{5.20}$$

These formulae enable us to evaluate (5.18). In doing so, we must realize that the displacements are small and, hence, that a linearization with respect to these displacements is allowed. In this way we approximate R^3 by

$$R^3 = R_0^3\left[1 + \frac{6a}{R_0^2}(u_1 - u_2)\right], \tag{5.21}$$

where

$$R_0 = R_0(z, \zeta) = \sqrt{4a^2 + (z - \zeta)^2} \geqslant 2a. \tag{5.22}$$

In the same way we linearize the numerator of the integrand in (5.18), thus finding an expression for $\mathbf{F}$ of the form

$$\mathbf{F}(z) = \mathbf{F}^{(0)}(z) + \mathbf{f}(z), \tag{5.23}$$

where $\mathbf{F}^{(0)}$ is independent of and $\mathbf{f}$ linear in the displacements. Hence, $\mathbf{F}^{(0)}$ is the force in the prebuckled state (causing the so called predeflections), which does not play any role in the determination of the buckling value for I_0. Therefore, we define $q(z)$ as the force per unit of length in the $\mathbf{e}_1$-direction acting on the deflected beam by

$$q(z) = (\mathbf{f}(z), \mathbf{e}_1), \tag{5.24}$$

and this force density is related to the deflection by the well-known beam equation

$$EI_y u^{iv}(z) = q(z). \tag{5.25}$$

The procedure described above yields the following expression for $q(z)$

$$q(z) = -\frac{\mu_0 I_0^2}{4\pi}\int_{-\infty}^{\infty}\left[\frac{u_1(z) - u_2(\zeta) - (z - \zeta)u_2'(\zeta)}{R_0^3} - \frac{12a^2(u_1(z) - u_2(\zeta))}{R_0^5}\right]\mathrm{d}\zeta. \tag{5.26}$$

For the further evaluation of (5.26) we assume symmetrical buckling, i.e. $u_1(z) = -u_2(z) =: u(z)$. After two partial integrations, in which it is used that $u(\zeta)$ is a periodic function in ζ, (5.26) becomes

$$q(z) = \frac{\mu_0 I_0^2}{4\pi a^2}\left[u(z) + a^2\int_{-\infty}^{\infty}\frac{u(\zeta) - u(z)}{R_0^3}\,\mathrm{d}\zeta\right]. \tag{5.27}$$

Finally, we realize that the second term in the right-hand side of (5.27) is $O(a^2/l^2)$ with respect to the first term. Since we have restricted ourselves to the cases $a \ll l$, we may neglect this term. Thus (5.25) takes the form

$$\frac{\mathrm{d}^4 u}{\mathrm{d}z^4} - \frac{\mu_0 I_0^2}{4\pi a^2 EI_y}u = 0. \tag{5.28}$$

The boundary conditions for $u(z)$ are (simply supported)

$$u(0) = u(l) = u''(0) = u''(l) = 0. \tag{5.29}$$

The first buckling mode satisfying these four boundary conditions is

$$u(z) = A \sin \pi z, \tag{5.30}$$

which after substitution into (5.28) (with $I_y = \pi R^4/4$) leads to the following result for the buckling value

$$\sqrt{\frac{\mu_0}{E}}\, I_0 = \pi a \left(\frac{\pi R}{l}\right)^2. \tag{5.31}$$

Table 4 shows that $1/\sqrt{Q_S} \approx m = a/R$ for m large (relative difference is less than 5% for $m \geqslant 4$) and then (5.17) becomes equal to (5.31). Hence, we conclude that for $a/R \geqslant 4$ the Biot-Savart approach presented here gives a good approximation for the buckling value. However, when the filaments come nearer to each other the correspondence becomes worse. In the limit $m \to 1$ the formula (5.31) gives a buckling value that is about 80% lower than the one according to (5.17).

We conclude with the remark that the results of Section 4 for two parallel, superconducting rods will be used in a forthcoming article [9], in which the buckling problem for two parallel toroidal superconductors is investigated. The fields for two rods, as found in the present paper, constitute a useful first approximation for the fields for two tori in case these tori are slender.

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